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On the Degree of Multivariate Bernstein Polynomial Operators*

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Let σ be a d-dimensional simplex with vertices $\mathbf{v}^0, \ldots, \mathbf{v}^d$ and $B_n(f, \cdot)$ denote the nth degree Bernstein polynomial of a continuous function f on σ. Dahmen and Micchelli (*Stud. Sci. Hungar.* **23** (1988), 265–287) proved that $B_n(f, \cdot) \geqslant B_{n+1}(f, \cdot)$, $n \in \mathbf{N}$, for any convex function f on σ, and it is clear that a necessary and sufficient condition for the inequality to become an identity for all $n \in \mathbf{N}$ is that f is an affine polynomial. Let σ_m be the mth simplicial subdivision of σ (which will be defined precisely later). By using a *degree-raising* formula, the result of Dahmen and Micchelli can be extended to $B_{mn}(f, \cdot) \geqslant B_{mn+1}(f, \cdot)$, $n \in \mathbf{N}$, for any f which is convex on every cell of σ_m. The objective of this paper is to derive conditions under which this inequality becomes an identity.

1. INTRODUCTION

As usual, let $\mathbf{R}$ denote the set of real numbers, $\mathbf{Z}_+$ the set of all nonnegative integers and $\mathbf{N} = \mathbf{Z}_+ \setminus \{0\}$. Thus, $\mathbf{R}^d$ is the d-dimensional Euclidean space and $\mathbf{Z}_+^d$ can be used as a multi-index set. Let σ be a d-dimensional simplex with vertex set $V = \{\mathbf{v}^0, \ldots, \mathbf{v}^d\}$. Here, we assume that $\mathbf{v}^i \in \mathbf{R}^d$, $i = 0, \ldots, d$, are in the general position, namely, the vectors $\mathbf{v}^i - \mathbf{v}^0$, $i = 1, \ldots, d$, are lineary independent. It is clear that, for any $\mathbf{x} \in \mathbf{R}^d$, there exists a unique $\xi = (\xi_0, \ldots, \xi_d) \in \mathbf{R}^{d+1}$ such that

$$\mathbf{x} = \sum_{i=0}^{d} \xi_i \mathbf{v}^i, \qquad \sum_{i=0}^{d} \xi_i = 1.$$

The coefficient $(d+1)$-tuple $\xi = (\xi_0, \ldots, \xi_d)$ is called the barycentric coordinates of $\mathbf{x}$ with respect to the simplex σ.

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Let $\alpha=(\alpha_0, ..., \alpha_d) \in \mathbf{Z}_+^{d+1}$ be a multi-index with

$$|\alpha| := \sum_{i=0}^{d} \alpha_i = n.$$

The Bernstein polynomial basis of degree n is given by

$$B_{\alpha,n}(\mathbf{x}) = \binom{n}{\alpha} \xi^\alpha, \qquad \mathbf{x} \in \sigma, \qquad |\alpha| = n,$$

with

$$\binom{n}{\alpha} = \frac{n!}{\alpha_0!\,\alpha_1! \cdots \alpha_d!}$$

and $\xi^\alpha = \xi_0^{\alpha_0} \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$. Clearly,

$$B_{\alpha,n}(\mathbf{x}) \geqslant 0 \qquad \text{for} \quad \mathbf{x} \in \sigma, \qquad \text{and} \qquad \sum_{|\alpha|=n} B_{\alpha,n}(\mathbf{x}) = 1.$$

Associated with any $f \in C(\sigma)$, the nth Bernstein polynomial of f on σ is defined by

$$B_n(f, \mathbf{x}) := \sum_{|\alpha|=n} f(\mathbf{x}_{\alpha,n})\, B_{\alpha,n}(\mathbf{x}),$$

where

$$\mathbf{x}_{\alpha,n} := \frac{1}{n} \sum_{i=0}^{d} \alpha_i \mathbf{v}^i, \qquad |\alpha| = n,$$

are called the nth B-net points of σ. Observe that there are $\binom{n+d}{d}$ nth B-net points on σ. Let $\mathbf{e}^i$, $i=1, ..., d$, denote the standard unit vectors in $\mathbf{R}^d$. In order to avoid an additional subscript or superscript, we will use $\mathbf{e}_0, ..., \mathbf{e}_d$ to denote the standard unit vectors in $\mathbf{R}^{d+1}$.

Recently, Chang and Davis [2] proved that

$$B_n(f, \cdot) \geqslant B_{n+1}(f, \cdot), \qquad n \in \mathbf{N},$$

for any convex function f on σ in the two-dimensional setting; Dahmen and Micchelli [4] extended this result to any $\mathbf{R}^d$. On the other hand, by the convergence property of $B_n(f, \cdot)$, it is easy to see that

$$B_n(f, \cdot) = B_{n+1}(f, \cdot), \qquad n \in \mathbf{N},$$

on σ if and only if f is an affine function on σ.

In order to extend this study to piecewise polynomials, we consider an mth simplicial subdivision σ_m of σ (which will be defined precisely in Section 2). Using a *degree-raising* formula, we have, for any $f \in C(\sigma)$,

$$B_{nm+1}(f, \cdot) - B_{nm}(f, \cdot) = \sum_{|\alpha| = nm+1} \left[f(\mathbf{x}_{\alpha, nm+1}) - \sum_{i=0}^{d} \frac{\alpha_i}{nm+1} f(\mathbf{x}_{\alpha - \mathbf{e}_i, nm}) \right] B_{\alpha, nm+1}(\cdot)$$

on σ. Since

$$\mathbf{x}_{\alpha, nm+1} = \sum_{i=0}^{d} \frac{\alpha_i}{nm+1} \mathbf{x}_{\alpha - \mathbf{e}_i, nm},$$

the assumption of convexity of f on each cell in σ_m yields

$$f(\mathbf{x}_{\alpha, nm+1}) - \sum_{i=0}^{d} \frac{\alpha_i}{nm+1} f(\mathbf{x}_{\alpha - \mathbf{e}_i, nm}) \leqslant 0.$$

Hence, we have the following result which is an extension of the polynomial result of Dahmen and Micchelli in [4] as stated above to piecewise polynomials.

THEOREM 1. *If f is convex on each cell in σ_m, then*

$$B_{nm}(f, \mathbf{x}) \geqslant B_{nm+1}(f, \mathbf{x}), \qquad \mathbf{x} \in \sigma,$$

and $n = 1, 2, \ldots$.

It is somewhat natural to believe that the inequality in Theorem 1 would become an identity if and only if $f \in S_1(\sigma_m)$, where $S_k(\sigma_m)$ denotes the space of continuous piecewise polynomials with total degree at most k on σ_m. The objective of this paper is to prove that indeed this statement holds. For the one-variable setting, this problem was already considered by Passow (see [7]). Our paper is organized as follows. In Section 2, we introduce a simplicial subdivision σ_m of the d-dimensional simplex σ and apply the *degree-raising* formula of Bernstein polynomials to derive a relation governing the coefficients for the identity $B_{nm}(f, \cdot) = B_{nm+1}(f, \cdot)$. The main results will be established in Section 3. We end this paper by proposing a conjecture for spline functions with total degree $k > 1$.

2. PRELIMINARIES

We begin by recalling some notations and terminologies. Observe that for $d = 2$, if the B-net points $\{\mathbf{x}_{\alpha, n}\}_{|\alpha| = n}$ on σ are considered as the vertices

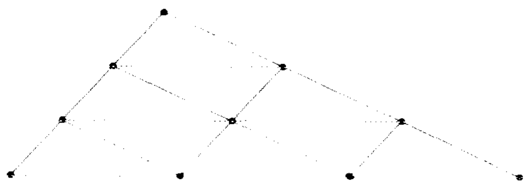

FIG. 1. Triangulation σ_3 ($d=2$, $n=3$).

of the subtriangles, then they form an nth triangulation σ_n of σ (see Fig. 1). The elements of σ_n have the same area and are actually similar to σ. Clearly, there are n^2 elements in σ_n. But for $d \geqslant 3$ the B-net points $\{\mathbf{x}_{\alpha,n}\}_{|\alpha|=n}$ do not give a complete simplicial subdivision as it can be seen in the following Figure 2, where $d=3$ and $n=2$. Nevertheless, according to [4], there is still a canonical way for constructing simplicial subdivisions of σ as follows, and this will allow us to apply an essential tool called "degree-raising argument". Let $\mathscr{P}$ be the set of all permutations of $\{1, 2, ..., d\}$ and define $\sigma_\pi \subset \mathbf{R}^d$ for $\pi \in \mathscr{P}$ via

$$\begin{aligned}\sigma_\pi &:= \{\mathbf{x} \in \mathbf{R}^d \colon 1 \geqslant x_{\pi(1)} \geqslant \cdots \geqslant x_{\pi(d)} \geqslant 0\}\\ &= [0, \mathbf{e}^{\pi(1)}, \mathbf{e}^{\pi(1)} + \mathbf{e}^{\pi(2)}, ..., \mathbf{e}^{\pi(1)} + \cdots + \mathbf{e}^{\pi(d)}].\end{aligned}$$

Clearly, the collection $\{\sigma_\pi \colon \pi \in \mathscr{P}\}$ forms a simplicial subdivision of the unit cube, and in addition it is also shown in [3] that $\mathscr{T} = \{\sigma_\pi + \alpha \colon \alpha \in \mathbf{Z}^d, \pi \in \mathscr{P}\}$ is a simplicial subdivision of $\mathbf{R}^d$. Let $\iota \in \mathscr{P}$ denote the identity, so that $\mathscr{T}_n = (\mathscr{T}/n) \cap \sigma_\iota$ forms a simplicial subdivision of σ_ι with vertices $\mathbf{v} = (v_0, ..., v_d) \in \mathbf{R}^d$ and nv_i are nonnegative integers with $1 \geqslant v_1 \geqslant \cdots \geqslant v_d$. Thus, for any affine map $A \colon \sigma_\iota \to \sigma$ and any $n \in \mathbf{N}$, the collection $\sigma_{A,n} = A(\mathscr{T}_n)$ forms an nth simplicial subdivision of σ. It is easy to see that there are n^d subsimplices in the nth subdivision $\sigma_{A,n}$ of σ. Let $\sigma_{A,n} = \{\hat{\sigma}_n^k\}_{k=1}^{n^d}$. We call the subsimplex $\hat{\sigma}_n^k$ a cell of $\sigma_{A,n}$. Since different choises of A only result in a permutation of the coordinates in σ, we will

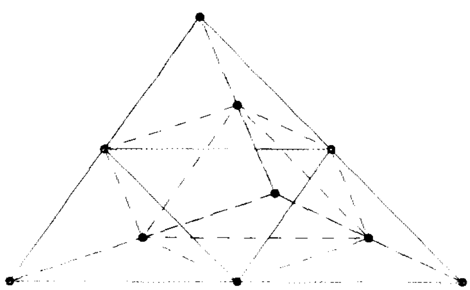

FIG. 2. Incomplete triangulation of a tetrahedron.

choose the same affine map A to form subdivisions of σ in the following discussion. For instance, we may restrict our consideration to the special case $A(0)=\mathbf{v}^0$ and $A(\mathbf{e}^1+\cdots+\mathbf{e}^k)=\mathbf{v}^k$, $k=1, 2, ..., d$, for which we will denote the nth subdivision of σ by σ_n. Here $\mathbf{e}^i$, $i=1, ..., d$, are the standard unit vectors in $\mathbf{R}^d$. For $d=2$, it can be verified that $\sigma_{A,n}$ is independent of A and agrees with the triangulation σ_n described earlier.

The following result is a consequence of the degree-raising formula for Bernstein polynomials. Since the proof is standard, we omit its proof and only refer the readers to [4].

LEMMA 1. *Let $f\in C(\sigma)$ and $n, m\in\mathbf{N}$. Then*

$$B_{nm}(f,\cdot)=B_{nm+1}(f,\cdot)$$

if and only if

$$f(\mathbf{x}_{\alpha,nm+1})=\sum_{i=0}^{d}\frac{\alpha_i}{nm+1}f(\mathbf{x}_{\alpha-\mathbf{e}_i,nm}) \tag{1}$$

for all $\alpha=(\alpha_0, ..., \alpha_d)\in\mathbf{Z}_+^{d+1}$ with $|\alpha|=nm+1$.

Remarks. 1. Here, we point out that even though $f(\mathbf{x}_{\alpha-\mathbf{e}_i,nm})$ may not be defined for $\alpha_i=0$ in (1), the corresponding coefficient $\alpha_i/(nm+1)$ is zero anyway. In this paper, we always assume that $\mathbf{x}_{\alpha-\mathbf{e}_i,n}$ makes sense; in other words, in case $\alpha_i=0$, we automatically delete the corresponding B-net point $\mathbf{x}_{\alpha-\mathbf{e}_i,n}$.

2. The restriction $B_n(f,\cdot)=B_{n+1}(f,\cdot)$ shows that the function values of f at the $(n+1)$st layer of B-net points $\mathbf{x}_{\alpha,n+1}$ is a convex combination of the values of f at some nth layer of B-net points, i.e.,

$$f(\mathbf{x}_{\alpha,n+1})=\sum_{i=0}^{d}\frac{\alpha_i}{n+1}f(\mathbf{x}_{\alpha-\mathbf{e}_i,n}), \tag{2}$$

where $\alpha=(\alpha_0, ..., \alpha_d)\in\mathbf{Z}_+^{d+1}$ with $|\alpha|=n+1$, and

$$\mathbf{x}_{\alpha,n+1}=\sum_{i=0}^{d}\frac{\alpha_i}{n+1}\mathbf{x}_{\alpha-\mathbf{e}_i,n}. \tag{3}$$

3. MAIN RESULTS

For $m\in\mathbf{Z}_+$, let $\pi_k(\sigma_m)$ denote the space of piecewise polynomial functions on σ_m with total degree at most k. Also, let $S_k(\sigma_m)=\pi_k(\sigma_m)\cap C(\sigma)$. In this section, we derive a characterization of f that satisfies

$$B_{nm}(f,\mathbf{x})=B_{nm+1}(f,\mathbf{x}),\qquad \mathbf{x}\in\sigma,\quad n\in\mathbf{N}.$$

For $f \in C(\sigma)$, we recall that

$$B_n(f, \cdot) = B_{n+1}(f, \cdot), \quad n \in \mathbf{N},$$

on σ if and only if $f \in S_1(\sigma)$. For $m \geq 2$, this problem becomes much more complicated. The following theorems, namely, Theorems 2–4, may be considered to be generalizations of the results of [7] to the d-dimensional setting. Let $\sigma(\mathbf{x}_{\alpha - \mathbf{e}_0, n}, ..., \mathbf{x}_{\alpha - \mathbf{e}_d, n})$ be the subsimplex with vertices $\mathbf{x}_{\alpha - \mathbf{e}_0, n}, ..., \mathbf{x}_{\alpha - \mathbf{e}_d, n}$. Our first result in this direction is the following.

THEOREM 2. *Let $f \in S_1(\sigma_m)$ and $n \in \mathbf{N}$. Then the total degree of the Bernstein polynomial $B_{mn+1}(f, \cdot)$ is at most mn. In particular,*

$$B_{mn+1}(f, \cdot) = B_{mn}(f, \cdot), \qquad n = 1, 2, ... \tag{4}$$

Proof. By Lemma 1, we have

$$B_{nm+1}(f, \cdot) - B_{nm}(f, \cdot)$$
$$= \sum_{|\alpha| = nm+1} \left[f(\mathbf{x}_{\alpha, nm+1}) - \sum_{i=0}^{d} \frac{\alpha_i}{nm+1} f(\mathbf{x}_{\alpha - \mathbf{e}_i, nm}) \right] B_{\alpha, nm+1}(\cdot). \tag{5}$$

We note that, for any B-net point $\mathbf{x}_{\alpha, nm+1}$, there are cells

$$\sigma(\mathbf{x}_{\alpha - \mathbf{e}_0, nm}, ..., \mathbf{x}_{\alpha - \mathbf{e}_d, nm})$$

in σ_{nm} with the vertex set $\{\mathbf{x}_{\alpha - \mathbf{e}_i, nm} : i = 0, 1, ..., d\}$ and $\hat{\sigma}_m^k$ in σ_m such that

$$\mathbf{x}_{\alpha, nm+1} \in \sigma(\mathbf{x}_{\alpha - \mathbf{e}_0, nm}, ..., \mathbf{x}_{\alpha - \mathbf{e}_d, nm}),$$
$$\sigma(\mathbf{x}_{\alpha - \mathbf{e}_0, nm}, ..., \mathbf{x}_{\alpha - \mathbf{e}_d, nm}) \subset \hat{\sigma}_m^k,$$

and for all $n \in \mathbf{N}$,

$$\mathbf{x}_{\alpha - \mathbf{e}_i, nm} \in \hat{\sigma}_m^k, \qquad i = 0, 1, ..., d. \tag{6}$$

By (3), we note that the barycentric coordinates of $\mathbf{x}_{\alpha, nm+1}$ with respect to the simplex $\sigma(\mathbf{x}_{\alpha - \mathbf{e}_0, nm}, ..., \mathbf{x}_{\alpha - \mathbf{e}_d, nm})$ is given by

$$\left(\frac{\alpha_0}{mn+1}, ..., \frac{\alpha_d}{mn+1} \right).$$

Because $f \in S_1(\sigma_m)$, it is an affine polynomial on any cell of the mth simplicial subdivision of σ. The linearity of f on $\hat{\sigma}_m^k$ shows that

$$f(\mathbf{x}_{\alpha, mn+1}) = \sum_{i=0}^{d} \frac{\alpha_i}{mn+1} f(\mathbf{x}_{\alpha - \mathbf{e}_i, mn}).$$

Hence, by applying (5), the conclusion follows. ∎

Next we consider a partial converse of Theorem 2 in the case $m > 1$. The full converse of Theorem 2 is still open even in the one-dimensional setting. We say that a function f is axially convex if it is convex in any direction parallel to the edges of the simplex σ (see [5] and the references therein), i.e.,

$$f(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) \leqslant tf(\mathbf{x}^1) + (1-t)\, f(\mathbf{x}^2)$$

holds for every $t \in [0, 1]$ and any $\mathbf{x}^1, \mathbf{x}^2$ such that $\mathbf{x}^1 - \mathbf{x}^2 = \theta(\mathbf{v}^i - \mathbf{v}^j)$, for some $0 \leqslant i < j \leqslant d$ and some $\theta \in \mathbf{R}$. The same argument in the proof of Theorem 1 also gives

$$B_{nm}(f, \mathbf{x}) \geqslant B_{nm+1}(f, \mathbf{x}), \qquad x \in \sigma, \quad n \in \mathbf{N},$$

whenever f is axially convex on each cell in σ_m.

We are now in a position to prove the following.

THEOREM 3. *Let $f \in C(\sigma)$ be axially convex in each cell in σ_m. If*

$$B_{mn+1}(f, \cdot) = B_{mn}(f, \cdot), \qquad n \in \mathbf{N},$$

then $f \in S_1(\sigma_m)$.

Proof. By the hypothesis and applying Lemma 1, we have

$$\sum_{\ell=0}^{d} \frac{\alpha_\ell}{mn+1}\, f(\mathbf{x}_{\alpha - \mathbf{e}_\ell,\, nm}) - f(\mathbf{x}_{\alpha,\, mn+1}) = 0,$$

for any $n \in \mathbf{N}$ and $\alpha \in \mathbf{Z}_+^{d+1}$ with $|\alpha| = mn + 1$. On the other hand, by (3), we have

$$\mathbf{x}_{\alpha,\, mn+1} = \sum_{\ell=0}^{d} \frac{\alpha_\ell}{mn+1} \mathbf{x}_{\alpha - \mathbf{e}_\ell,\, nm}.$$

This shows that the point $(\mathbf{x}_{\alpha,\, mn+1}, f(\mathbf{x}_{\alpha,\, mn+1}))$, which is on the surface $y = f(\mathbf{x})$, also lies on the graph of the affine function

$$L(\mathbf{x}) = \sum_{\ell=0}^{d} \xi_\ell f(\mathbf{x}_{\alpha - \mathbf{e}_\ell,\, nm}),$$

with $\mathbf{x} \in \sigma(\mathbf{x}_{\alpha - \mathbf{e}_0,\, mn}, ..., \mathbf{x}_{\alpha - \mathbf{e}_d,\, mn})$ and $(\xi_0, ..., \xi_d)$ the barycentric coordinates of $\mathbf{x}$ with respect to the cell $\sigma(\mathbf{x}_{\alpha - \mathbf{e}_0,\, nm}, ..., \mathbf{x}_{\alpha - \mathbf{e}_d,\, nm})$, $n = 1, 2, ...$. The axial convexity and continuity of f guarantee that f is an affine polynomial on each cell in σ_m, so that $f \in S_1(\sigma_m)$. ∎

It is clear that convexity is a stronger condition than axial convexity. For example, the function $f(x, y) = -xy$ is axially convex but not convex on $\sigma = \{(x, y): x + y \leqslant 1,\ x, y \geqslant 0\}$. Hence, the following conclusion holds.

COROLLARY 1. *Let $f \in C(\sigma)$ such that the restriction of f on each cell in σ_m is either convex or concave, and*

$$B_{nm+1}(f, \mathbf{x}) = B_{nm}(f, \mathbf{x}), \qquad \mathbf{x} \in \sigma, \quad n \in \mathbf{N}.$$

Then $f \in S_1(\sigma_m)$.

Let $D(\sigma_m)$ denote the set of net points (or vertices) of the mth subdivision σ_m of σ. We also have the following.

THEOREM 4. *Let σ_q be a simplicial subdivision of the simplex σ, $D(\sigma_q)$ the set of net points of σ_q, $m \in \mathbf{N}$, and $f \in S_1(\sigma_q)$ such that*

$$B_{nm}(f, \mathbf{x}) = B_{nm+1}(f, \mathbf{x}), \qquad \mathbf{x} \in \sigma, \quad n \in \mathbf{N}.$$

Then $D(\sigma_q) \subset D(\sigma_m)$.

Proof. Suppose that σ^* is an arbitrary cell in σ_m and there exists some $\mathbf{x}^* \in D(\sigma_q) \cap \operatorname{Int}(\sigma^*)$. In addition, suppose that there are two d-dimensional subsimplices σ_q^1 and σ_q^2 in σ_q that have a common vertex $\mathbf{x}^*$ and a common $(d-1)$-dimensional simplex, and that the planar surfaces defined by the restrictions of $f \in S_1(\sigma_q)$ on σ_q^1 and σ_q^2 have different normal vectors. Then we can find a neighborhood $N(\mathbf{x}^*)$ of $\mathbf{x}^*$ such that $N(\mathbf{x}^*) \subset \operatorname{Int}(\sigma^*)$ and an open set $O \subset N(\mathbf{x}^*) \cap (\sigma_q^1 \cup \sigma_q^2)$ such that $O \cap \sigma_q^i \neq \emptyset$, $i = 1, 2$. Obviously, f is convex (or concave) in O since it is piecewise linear; and so, for sufficiently large n, there exists a $(d+1)$-dimensional array

$$K_0\colon \{\mathbf{x}_{\beta - \mathbf{e}_0, mn}, \ldots, \mathbf{x}_{\beta - \mathbf{e}_d, nm}\} \subset O$$

for some points $\beta \in \mathbf{Z}_+^{d+1}$ with $|\beta| = nm + 1$, and only some of the points in K_0, say $\mathbf{x}_{\beta - \mathbf{e}_0, nm}, \ldots, \mathbf{x}_{\beta - \mathbf{e}_j, nm}$ lie in $O \cap \sigma_q^1$, and the others are in $O \cap \sigma_q^2$.

Let us introduce an affine function

$$g(\mathbf{x}) = \sum_{\ell=0}^{d} \xi_\ell f(\mathbf{x}_{\beta - \mathbf{e}_\ell, nm}),$$

where

$$\mathbf{x} = \sum_{\ell=0}^{d} \xi_\ell \mathbf{x}_{\beta - \mathbf{e}_\ell, nm}, \qquad 0 \leqslant \xi_\ell \leqslant 1, \qquad \sum_{\ell=0}^{d} \xi_\ell = 1.$$

Since f is convex (or concave) on O and the planar surfaces defined by the restrictions of f on $O \cap \sigma_q^1$ and $O \cap \sigma_q^2$ have different normal vectors, we have, for any $\mathbf{x} \in \sigma(\mathbf{x}_{\beta - \mathbf{e}_0, nm}, \ldots, \mathbf{x}_{\beta - \mathbf{e}_d, nm})$,

$$f(\mathbf{x}) < g(\mathbf{x}) \qquad (\text{or } f(\mathbf{x}) > g(\mathbf{x})).$$

Therefore, without loss of generality, we may assume that f is convex. By (3) and (6), we have

$$f(\mathbf{x}_{\beta, mn+1}) < \sum_{\ell=0}^{d} \frac{\beta_\ell}{mn+1} f(\mathbf{x}_{\beta - \mathbf{e}_\ell, nm}).$$

On the other hand, by the assumption

$$B_{mn+1}(f, \mathbf{x}) = B_{mn}(f, \mathbf{x})$$

and Lemma 1, we obtain

$$f(\mathbf{x}_{\beta, mn+1}) = \sum_{\ell=0}^{d} \frac{\beta_\ell}{mn+1} f(\mathbf{x}_{\beta - \mathbf{e}_\ell, nm}).$$

This contradiction shows that

$$\mathbf{x}^* \notin \mathrm{Int}(\sigma^*).$$

Furthermore, since the Bernstein polynomial on the boundary $\partial\sigma$ could be obtained by restricting $B_n(f, \cdot)$ to $\partial\sigma$, Lemma 1 still holds even if we restrict ourselves to the boundary of σ. So, applying the same argument to $\partial\sigma$, we may conclude that $\mathbf{x}^*$ is not in the relative interior of $\partial\sigma^*$. By repeating this procedure on the lower dimensional boundaries, we have $\mathbf{x}^* \in D(\sigma_m)$. This completes the proof of Theorem 4. ▮

It is natural to ask the possibility of extending our results to $S_k(\sigma_m)$, $k > 1$. In this regard, we believe that Theorem 3 holds mainly because of the affine polynomial reproduction property of the Bernstein operator $B_n(f, \cdot)$. Let us consider certain linear combinations of Bernstein polynomials introduced first by Butzer [1] in the univariate case and by Wu [8] in the multidimensional setting, for reproducing polynomials $p \in \pi_k$. More precisely, let $L_n^{(0)} = B_n$, and define $L_n^{(k)}$ recursively by

$$L_n^{(k)} = (2^k - 1)^{-1} (2^k L_{2n}^{(k-1)} - L_n^{(k-1)}), \qquad k = 1, 2, \ldots.$$

Then

$$L_n^{(k)} p = p \qquad \forall p \in \pi_{k+1},$$

(see [6, 8]). An extension to $S_k(\sigma_m)$ can be formulated as follows.

Conjecture. Let σ_m be the mth simplicial subdivision of σ, $m \in \mathbf{N}$, and k be any positive integer. Then

$$f \in S_k(\sigma_m) \cap C(\sigma)$$

if and only if

$$\Delta^{k+1} L_{nm}^{(k)}(f, \cdot) = 0, \qquad n \in \mathbf{N},$$

where Δ is the difference operator defined by

$$\Delta L_n = L_{n+1} - L_n,$$

and $\Delta^{k+1} = \Delta^k \Delta$.

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